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Patterns of Duality in N=1 SUSY Gauge Theories

or: *Seating Preferences of Theatre-Going Non-Abelian Dualities*

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We study the patterns in the duality of a wide class of N=1 supersymmetric gauge theories in four dimensions. We present many new generalizations of the classic duality models of Kutasov and Schwimmer, which have themselves been generalized numerous times in works of Intriligator, Leigh and the present authors. All of these models contain one or two fields in a two-index tensor representation, along with fields in the defining representation. The superpotential for the two-index tensor(s) resembles A_k or D_k singularity forms, generalized from numbers to matrices. Looking at the ensemble of these models, classifying them by superpotential, gauge group, and “level” — for terminology we appeal to the architecture of a typical European-style theatre — we identify emerging patterns and note numerous interesting puzzles.

1. Introduction

At present the known duality transformations in four-dimensional N=1 supersymmetric gauge theories are few, and generally fall into a few special classes. The first examples originated in the work of Seiberg [1] and were elaborated in Refs. [2] and [3]. These dualities exchange two theories with a single gauge group $SU(N)$, $SO(N)$ or $Sp(N)$, and matter in the defining representation of the gauge group (fundamental plus anti-fundamental for SU , vector for SO , fundamental for Sp .) Certain gauge singlets are also required.

In the models of Ref. [4], a field X in the adjoint representation is added to the SU duality of Ref. [1]. This field is given a superpotential $\text{Tr}X^{k+1}$, corresponding to an A_k singularity under the usual ADE classification [5]. The dual theory also has an adjoint field \bar{X} with superpotential $\text{Tr}\bar{X}^{k+1}$. The duality reduces to that of [1] in the case $k = 1$. This A_k singularity structure can be generalized to a wide variety of other gauge groups and matter content, as in Refs. [6,7,8].

Additional and unrelated dualities, involving spinors of *Spin* groups and their subgroups, were found in [9], and were shown to flow to the SO duality of Refs. [1,2]. Other claims of duality transformations have been made in [10,11,12,13]; many of these follow from the ones listed above.

Recently, one of us presented [14] a new generalization of Ref. [4]. Instead of one adjoint field X with an A_k superpotential, two adjoint fields X and Y were introduced and were given a D_{k+2} superpotential of the form

$$W = \text{Tr}X^{k+1} + \text{Tr}XY^2. \quad (1.1)$$

It was shown that for k odd this model has the special properties which have appeared in all A_k dualities (namely, its chiral ring is truncated) and is dual to a model of similar type.

Furthermore, Ref. [14] also presented a duality in which the gauge group is $SU(N_c) \times SU(N'_c)$, with a field F in the $(\mathbf{N}_c, \mathbf{N}'_c)$ representation, another \tilde{F} in the conjugate representation, fields X_1 and X_2 in the adjoint representations of the two group factors, and for each group factor a number of flavors in its fundamental and anti-fundamental representations. The superpotential is

$$W = \text{Tr}X_1^{k+1} + \text{Tr}X_2^{k+1} + \text{Tr}X_1\tilde{F}F - \text{Tr}X_2\tilde{F}F \quad (1.2)$$

and again the theory is dual to a model of similar type.

In this paper, we present a number of generalizations of the two classes of dual pairs presented in Ref. [14]. We quickly review the A_k models and outline the pattern of the D_{k+2} models. After touching on the key points of the D_{k+2} model of Ref. [14], we examine new D_{k+2} models with $SO(N_c)$ gauge symmetry with two symmetric tensors and $Sp(N_c)$ gauge symmetry with two anti-symmetric tensors. We review the $SU(N_c) \times SU(N'_c)$ duality of Ref. [14], and derive it from the D_{k+2} model using an improved and simplified method. We then generalize it to $SO \times SO$ and $Sp \times Sp$ groups. Next, we examine a large number of generalizations of the D_{k+2} class, involving SU , SO and Sp gauge groups. We consider various representations for the field Y , but with X always in the adjoint (symmetric) [anti-symmetric] representation for SU (SO) [Sp] groups. In each of these cases, the D_3 models are the same as the A_3 models of Refs. [4,6,7,8]. Finally, we discuss the overall structure of the known examples and list puzzles which remain unsolved at the present time.

2. Review of A_k Models

We begin by reviewing the results of Refs. [4,6,7,8]. In that series of papers, many classes of A_k -type models were exhibited; a summary of those nineteen models is given in Sec. 2.1–2.19 of [8]. They can be organized into the multi-level structure of fig. 1 and fig. 2. For ease of reference we have chosen to refer to them using the terminology for the levels of a standard European-style theatre.

The orchestra models [4,6] have a superpotential of the form $\text{Tr}X^{k+1}$, where X is in the adjoint (symmetric) [anti-symmetric] representation for SU (SO) [Sp] groups. In addition each model has a number of fields Q in the defining representation of the gauge group, transforming under a flavor symmetry. Each model is dual to a theory of similar type, containing in addition certain fields M which are singlets of the gauge group but non-singlets of the flavor symmetry. The dual superpotential for the dual field \bar{X} is again $\text{Tr}\bar{X}^{k+1}$ and, in addition, contains terms coupling the mesons M to the dual fields q and \bar{X} .

The mezzanine models [7,8] differ from the orchestra models in that they have a superpotential $\text{Tr}X^{2(k+1)}$ for SO (Sp) with X in the anti-symmetric (symmetric) representation, or $\text{Tr}(X\tilde{X})^{k+1}$ for SU with X in the symmetric or anti-symmetric representation and \tilde{X} in its conjugate.

The balcony model, which is chiral, has gauge group SU and a superpotential $\text{Tr}(X\tilde{X})^{2(k+1)}$, where X is in the symmetric and \tilde{X} is in the conjugate anti-symmetric representation.

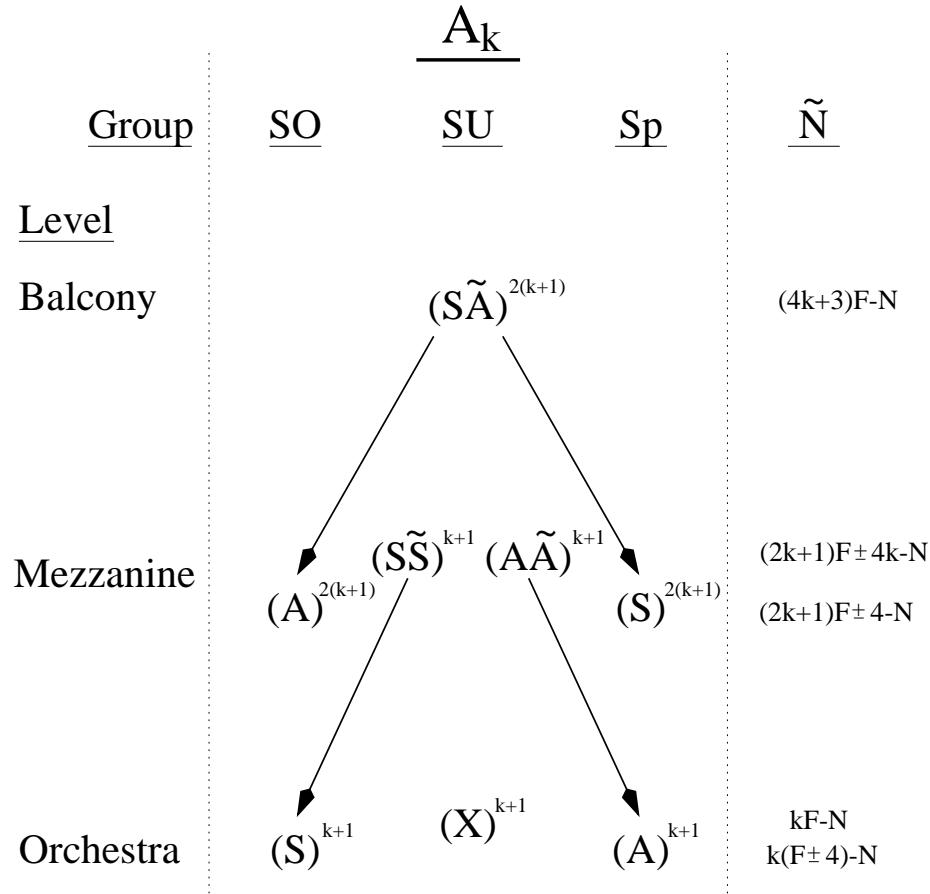


Fig. 1: The simplest A_k models of Refs. [4,6,7,8], organized as suggested by Sec. 2 of [8]. The notation in the figure is as follows: X represents a field in the adjoint representation of $SU(N)$, S and A represent symmetric and anti-symmetric tensor representations, and a tilde represents a conjugate representation of $SU(N)$. For SU [SO] (Sp) groups, $N = N_c$ [N_c] ($2N_c$) and $F = N_f$ [N_f] ($2N_f$). The superpotential for each model is given. Each model is dual to a model of similar type, with color group $SU(\tilde{N})$, $SO(\tilde{N})$ or $Sp(\frac{\tilde{N}}{2})$. The arrows indicate that there are flat directions along which the upper models flow to the lower ones.

These models and their interrelations are shown in fig. 1. Notice the many patterns in the dual color groups, listed at the right of the figure. The duality transformations are invariant under simultaneous exchange of SO and Sp groups and symmetric and anti-symmetric tensors, along with $N \leftrightarrow -N$, $\tilde{N} \leftrightarrow -\tilde{N}$, and $F \leftrightarrow -F$. The group theory behind this \mathbb{Z}_2 symmetry has been discussed in the literature [15].

Some of these models have flat directions along which they flow to others in the infrared, as indicated by the arrows in fig. 1. An important check on the conjectured duality transformations is that they commute with these flows [8] in a highly non-trivial way.

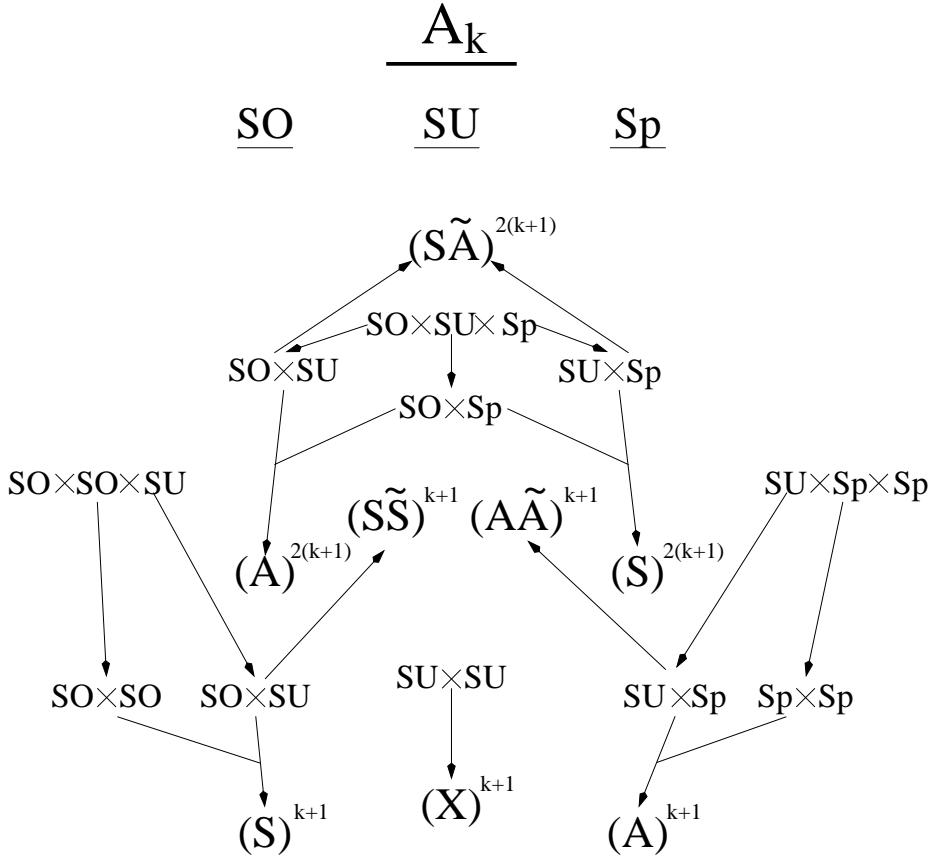


Fig. 2: The models of the previous figure along with the loge models of Ref. [8]. The loge models consist of a product of two or three gauge groups, which is displayed on the figure. Each is dual to a model of similar type, with the same groups. The process of confinement of one group factor causes the loge models to flow along the arrows in the diagram to models with one fewer group factor. More generally, duality of a single group factor relates the duality of each loge model to a duality of a model with one fewer group factor, along the same arrows.

Related to each of these models are the loge models [8] shown in fig. 2 in which the fields X or \tilde{X} are deconfined. An adjoint representation of $SU(N)$ can always be deconfined [1,10,8,11] using an $SU(N) \times SU(N-1)$ group with a field F in the $(\mathbf{N}, \mathbf{N-1})$ and a field \tilde{F} in its conjugate. Symmetric [anti-symmetric] representations of a group G can always be deconfined using $SO(N) \times SO(N+4)$ [$Sp(N) \times Sp(N-2)$] groups with a single field in the $(\mathbf{N}, \mathbf{N+4})$ [$(\mathbf{N}, \mathbf{N-2})$]. Under this deconfinement the degree of the superpotential is increased, since each field X is rewritten as FF or $F\tilde{F}$. What is remarkable is that, in contrast to various other examples of deconfinement which appear in the literature [11], the dualities of Refs. [4,6,7,8,9] are preserved when deconfinement is applied to both sides; a complete closed diagram of theories, in which two dual loge models flow to two dual

orchestra, mezzanine, or balcony models, is generated.

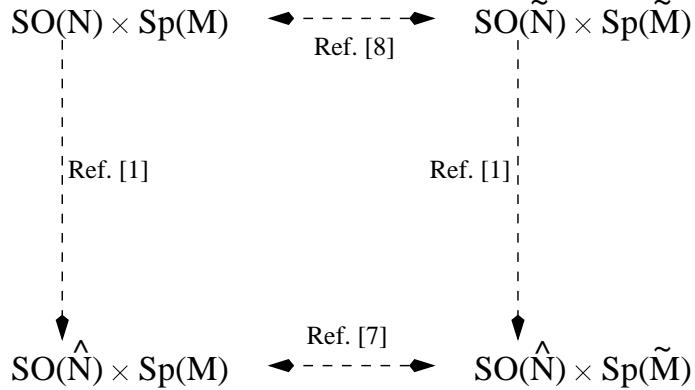


Fig. 3: A closure diagram for a dual pair of theories. This is a crucial consistency test of any claim of duality. The two gauge groups at the top of the diagram are supposed to describe the same theory in the infrared, Ref. [8]. This should be true for any choice of gauge couplings. Take the coupling of $SO(N)$ to be much larger than that of $Sp(M)$; the duality requires the same for $SO(\tilde{N})$. A strongly coupled $SO(N)$ theory is better described by its dual; here the appropriate dual is that of Ref. [1]. The low-energy $SO(\hat{N})$ theories have the same color group and are weakly coupled spectators to the Sp groups, which fortunately are related by a duality of Ref. [7].

An example of this process is illustrated in fig. 3. Consider the theory built on $SO(N) \times Sp(M)$, with a field F in the $(\mathbf{N}, \mathbf{2M})$, N_f fields Q in the vector representation of $SO(N)$ and M_f fields in the fundamental representation of $Sp(M)$. The superpotential is $\text{Tr}F^{4(k+1)}$. As described in Sec. 2.12 of [8], the magnetic theory is similar, with gauge group $SO(\tilde{N}) \times Sp(\tilde{M})$ where

$$\begin{aligned}\tilde{N} &= 2(k+1)(M_f + N_f) - N_f - 2M ; \\ 2\tilde{M} &= 2(k+1)(M_f + N_f) - M_f - N .\end{aligned}\tag{2.1}$$

If the $SO(N)$ group becomes strongly coupled at an energy scale where the $Sp(M)$ group is still weakly coupled, then a better description of the theory below that scale is given by using the dual of the $SO(N)$ group (which has $N_f + 2M$ vector representations) with respect to the duality of Seiberg [1,2]. The theory then has gauge group $SO(\hat{N}) \times Sp(M)$, where $\hat{N} = N_f + 2M - N + 4$, and the Sp factor now has a symmetric field $S = F^2$ and a field f in the $(\hat{\mathbf{N}}, \mathbf{2M})$ with superpotential $\text{Tr}S^{2(k+1)} + Sff$, along with M_f fundamentals, which makes it an A_k mezzanine theory. Meanwhile, in the dual $SO(\tilde{N}) \times Sp(\tilde{M})$ theory,

the $SO(\tilde{N})$ theory becomes strongly coupled first and is also best represented by its dual under the duality of Seiberg. The low-energy theory is then $SO(\hat{\tilde{N}}) \times Sp(\tilde{M})$, where $\hat{\tilde{N}} = M_f + 2\tilde{M} - \tilde{N} + 4 = \hat{N}$, from (2.1). The Sp factor is now also a mezzanine theory. One may check, counting carefully the number of fundamentals in the Sp groups and accounting for the Sff term in the superpotential, that the $Sp(M)$ and $Sp(\tilde{M})$ factors are indeed dual under the mezzanine duality of [7].

If instead the Sp factors had become strongly coupled before the SO factors, the low-energy duality would have involved the SO mezzanine models.

Since each of the A_k dualities relates two theories of similar type, there are choices of N_f and N_c such that the theories are self-dual; that is, the electric and magnetic theories have the same gauge group and charged matter, though they will differ through the presence or absence of gauge singlet fields. The self-dual theories of Seiberg [1] have exactly marginal operators which take the form of meson mass terms and which are deeply connected with the duality transformation [16]. The presence of marginal operators in the self-dual models of the A_k series was noted in a few examples [4,7] and shown to be ubiquitous in Ref. [8]. The precise connection with the duality transformation is as yet unknown.

3. A Program Guide for the D_{k+2} Models

We present here a preview of the models we found that have a superpotential of the form D_{k+2} . We have organized them, like the A_k models, into a multi-level structure (see fig. 4).

For all the models, we have a superpotential of the form $\text{Tr}X^{k+1} + \text{Tr}XY^2$ where X is in the traceless adjoint (symmetric) [anti-symmetric] representation for SU (SO) [Sp] groups. In the orchestra models, the field Y is also in the traceless adjoint (symmetric) [anti-symmetric] representation for SU (SO) [Sp] groups. In addition each model has a number of fields Q in the defining representation of the gauge group, transforming under a flavor symmetry. Each model is dual to a theory of similar type, containing in addition certain fields M which are singlets of the gauge group but non-singlets of the flavor symmetry. The dual superpotential for the dual fields \bar{X}, \bar{Y} is again $\text{Tr}\bar{X}^{k+1} + \text{Tr}\bar{X}\bar{Y}^2$ and, in addition, contains terms coupling the mesons M to the dual fields q, \bar{X} and \bar{Y} .

The mezzanine models differ from the orchestra models in that Y is in the anti-symmetric (symmetric) representation for SO (Sp), while for SU the field Y is in the

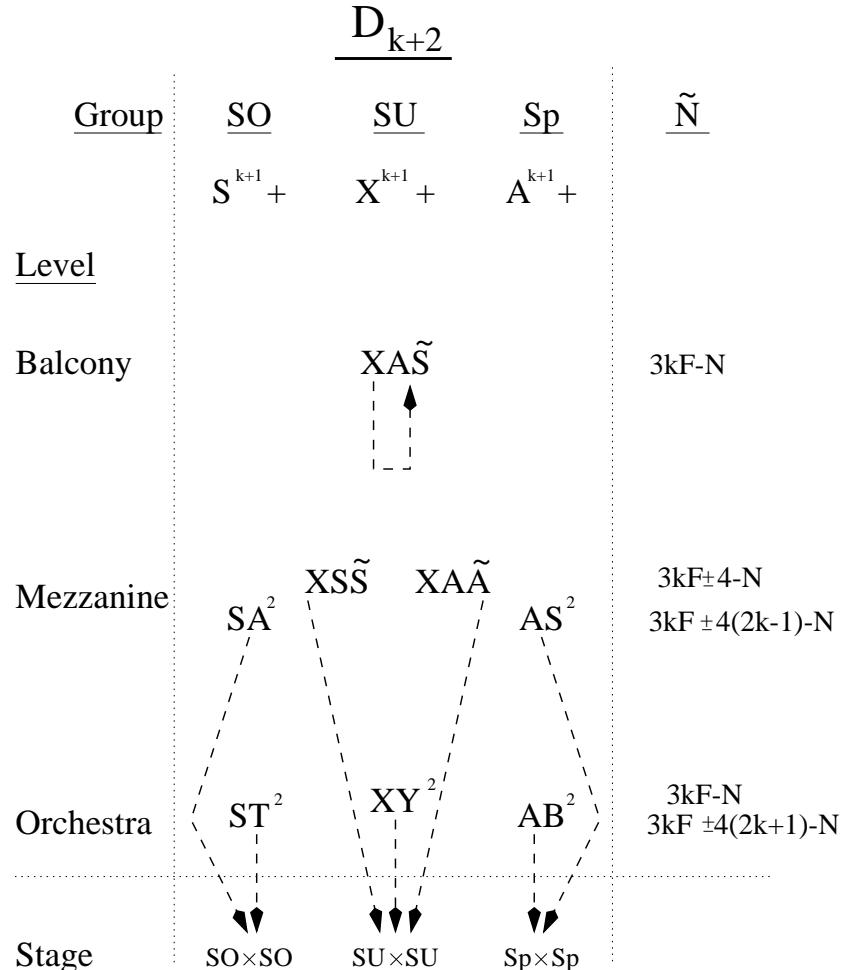


Fig. 4: The ensemble of D_{k+2} models described in this paper, along with the product models to which they are related. Notation is similar to fig. 1, with X, Y adjoint fields of $SU(N)$, S, T symmetric tensors and A, B anti-symmetric tensors, and a tilde representing a conjugate representation of $SU(N)$. For SU [SO] (Sp) groups, $N = N_c$ [N_c] ($2N_c$) and $F = N_f$ [N_f] ($2N_f$). The superpotential for each model is given, with the first term listed at the top and the second at the position of the model in the diagram; thus the superpotential for the Mezzanine $SO(N)$ model is $S^{k+1} + SA^2$, etc. Each model is dual to a model of similar type, with color group $SU(\tilde{N})$, $SO(\tilde{N})$ or $Sp(\frac{\tilde{N}}{2})$. The dashed arrows indicate that under certain perturbations the D_{k+2} models flow to stage models, except for the balcony model which flows to copies of itself with lower k .

anti-symmetric or symmetric representation and \tilde{Y} is in its conjugate. In the latter case the superpotential is $W = \text{Tr}X^{k+1} + \text{Tr}XY\tilde{Y}$.

The balcony model, which is chiral, has gauge group SU with fields Y in the anti-symmetric and \tilde{Y} in the conjugate symmetric representation; again the superpotential is $W = \text{Tr}X^{k+1} + \text{Tr}XY\tilde{Y}$.

In the A_k models, we have examples of dualities involving product gauge groups, which we call the loge models. The dynamics of these models can cause them to flow in the infrared to the A_k models of fig. 1, as shown in fig. 2. We have not found analogues of the loge models for D_{k+2} . Instead, we have found other models with product gauge groups, generalizing that of Ref. [14], with superpotentials in the form of Eq. (1.2). We will refer to these as stage models. The D_{k+2} orchestra and mezzanine models flow to the stage models under perturbation, as shown by the arrows in fig. 4, preserving the duality.

As in the A_k models, all of these theories are dual to models of similar type. As before there is a symmetry under which N , \tilde{N} and F change sign, SO and Sp groups are interchanged, and symmetric and anti-symmetric tensors representations are exchanged. Also, each of the models has choices of N_f and N_c for which it is self-dual and for which there are exactly marginal operators in the form of meson mass terms.

4. The D_{k+2} Orchestra Models

4.1. Orchestra: $SU(N_c)$ with Two Adjoint Tensors

In Ref. [14] a D_{k+2} generalization of the A_k models of [4] was found for odd k . The theory, with gauge group $SU(N_c)$, has the following matter content:

	$SU(N_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
Q	\mathbf{N}_c	\mathbf{N}_f	$\mathbf{1}$	1	$1 - \frac{N_c}{N_f(k+1)}$
\tilde{Q}	$\overline{\mathbf{N}_c}$	$\mathbf{1}$	$\overline{\mathbf{N}_f}$	-1	$1 - \frac{N_c}{N_f(k+1)}$
X	$\mathbf{N}_c^2 - \mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	$\frac{2}{k+1}$
Y	$\mathbf{N}_c^2 - \mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	$\frac{k}{k+1}$

The superpotential is

$$W = \frac{\text{Tr}X^{k+1}}{k+1} + \text{Tr}XY^2 + \lambda_1 \text{Tr}X + \lambda_2 \text{Tr}Y \quad (4.1)$$

where λ_1 and λ_2 are Lagrange multipliers which enforce the tracelessness condition on X and Y . The conditions for a supersymmetric vacuum that follow from this superpotential are

$$\begin{aligned} X^k + Y^2 + \lambda_1 &= 0 \\ XY + YX + \lambda_2 &= 0. \end{aligned} \quad (4.2)$$

These equations truncate the chiral ring for odd values of k . To illustrate this we can ignore λ_1 and λ_2 ; statements below will then be correct modulo lower order terms already included in the chiral ring. We can multiply X^k by Y from the right or left, and use the first equation in (4.2) to show that

$$YX^k + X^kY = -2Y^3. \quad (4.3)$$

Now, we can use the second equation in (4.2) to anticommute the Y fields through the X fields.

$$((-1)^k + 1)X^kY = -2Y^3 \quad (4.4)$$

Thus for odd k , $Y^3 = 0$. The chiral ring is now said to be truncated since it depends only on k and not on the size of the gauge group. The gauge invariant mesons are

$$\tilde{Q}_s X^j Y^l Q^r \quad (4.5)$$

where r and s are flavor indices and where the truncation requires $j = 0, 1, \dots, k-1$ and $l = 0, 1, 2$. These mesons are all in the $(\mathbf{N_f}, \overline{\mathbf{N_f}})$ representation of the flavor group.

This theory has a dual description in terms of an $SU(3kN_f - N_c)$ gauge theory with the following charged matter fields

	$SU(\tilde{N}_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
q	$\tilde{\mathbf{N}}_c$	$\overline{\mathbf{N_f}}$	$\mathbf{1}$	$\frac{N_c}{\tilde{N}_c}$	$1 - \frac{\tilde{N}_c}{N_f(k+1)}$
\tilde{q}	$\overline{\tilde{\mathbf{N}}_c}$	$\mathbf{1}$	$\mathbf{N_f}$	$-\frac{N_c}{\tilde{N}_c}$	$1 - \frac{\tilde{N}_c}{N_f(k+1)}$
\bar{X}	$\tilde{\mathbf{N}}_c^2 - \mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	$\frac{2}{k+1}$
\bar{Y}	$\tilde{\mathbf{N}}_c^2 - \mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	$\frac{k}{k+1}$

Throughout this paper, \tilde{N}_c will be used to denote the dual gauge group. The dual description also has gauge singlets, $(M_{jl})_s^r$, which are in a one-to-one mapping with the mesons in equation (4.5). The dual superpotential has the form

$$W = \frac{\text{Tr} \bar{X}^{k+1}}{k+1} + \text{Tr} \bar{X} \bar{Y}^2 + \sum_{j=0}^{k-1} \sum_{l=0}^2 M_{jl} \tilde{q} \bar{X}^{k-j-1} \bar{Y}^{2-l} q + \lambda_1 \text{Tr} \bar{X} + \lambda_2 \text{Tr} \bar{Y}. \quad (4.6)$$

where flavor indices are contracted in the obvious way.

If we consider the case $k = 1$ in (4.1), we see that the fields X and \bar{X} are massive and can be integrated out in the IR. The theories develop a quartic superpotential for Y ,

\bar{Y} , and the low-energy theories are (marginal deformations of) models known to be dual through the work of Kutasov and Schwimmer [4]. This is a manifestation of the equivalence between D_3 and A_3 singularities (though in this matrix generalization of singularity theory the equivalence is not exact.)

The duality is consistent along certain flat directions of this theory. Consider the superpotential (4.1) where k is taken to be odd. The minima of the superpotential (4.2) allow for an $SU(2n + km)$ gauge theory to have a flat direction along which X and Y get vacuum expectation values

$$\langle X \rangle = a \begin{pmatrix} 0_n & 0 & 0 & 0 & 0 & 0 & \cdot & 0 \\ 0 & 0_n & 0 & 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 1_m & 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & \omega_m & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & 0 & \omega_m^2 & 0 & \cdot & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega_m^3 & \cdot & 0 \\ \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \omega_m^{k-1} \end{pmatrix} \quad (4.7)$$

$$\langle Y \rangle = b \begin{pmatrix} 1_n & 0 & 0 & 0 & 0 & 0 & \cdot & 0 \\ 0 & -1_n & 0 & 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0_m & 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & 0_m & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & 0 & 0_m & 0 & \cdot & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_m & \cdot & 0 \\ \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdot & 0_m \end{pmatrix}$$

where $\omega = \exp \frac{2\pi i}{k}$ and $a^k = b^2 = \lambda_1$. (A subscript n on an entry indicates that it is proportional to the $n \times n$ dimensional unit matrix.) The expectation value for Y breaks the theory into $SU(n) \times SU(n) \times SU(km) \times U(1)^2$; that of the X field breaks it down to $SU(n) \times SU(n) \times SU(m)^k \times U(1)^{k+1}$. The fields X and Y each decompose into $k+2$ adjoints and fields in the (\mathbf{n}, \mathbf{n}) , (\mathbf{n}, \mathbf{m}) , (\mathbf{m}, \mathbf{n}) , and (\mathbf{m}, \mathbf{m}) representations. The field from X in the (\mathbf{n}, \mathbf{n}) representation, which we call F , and its conjugate \tilde{F} , do not receive a mass, owing to the tracelessness condition on X . The leading term in the low-energy superpotential is

$$W_L = \frac{\text{Tr}(F\tilde{F})^{\frac{k+1}{2}}}{k+1} . \quad (4.8)$$

All the other matter fields from X and Y are massive along this flat direction. The same expectation values in the dual theory give a similar result; the $SU(3kN_f - N_c)$ gauge group

is broken to $SU(kN_f - n) \times SU(kN_f - n) \times SU(N_f - m)^k \times U(1)^{k+1}$. The $SU(n) \times SU(n)$ factors of the low-energy theories are related under an A_k loge duality discussed in Sec. 2.11 of Ref. [8]; the k $SU(m)$ factors are simple SQCD models and are related under Seiberg's duality [1].

4.2. A New D_{k+2} Duality: $SO(N_c)$ with Two Symmetric Tensors

We consider a theory with $SO(N_c)$ gauge group and the following matter content:

	$SO(N_c)$	$SU(N_f)$	$U(1)_R$
Q	\mathbf{N}_c	\mathbf{N}_f	$1 - \frac{N_c - 4k - 2}{N_f(k+1)}$
X	$\frac{\mathbf{N}_c(\mathbf{N}_c+1)}{2} - \mathbf{1}$	$\mathbf{1}$	$\frac{2}{k+1}$
Y	$\frac{\mathbf{N}_c(\mathbf{N}_c+1)}{2} - \mathbf{1}$	$\mathbf{1}$	$\frac{k}{k+1}$

The superpotential for the theory is

$$W = \frac{\text{Tr}X^{k+1}}{k+1} + \text{Tr}XY^2 + \lambda_1 \text{Tr}X + \lambda_2 \text{Tr}Y \quad (4.9)$$

where k is odd. The vacua of the theory are given by

$$X^k + Y^2 + \lambda_1 = 0 = XY + YX + \lambda_2 . \quad (4.10)$$

As in the previous case, these equations truncate the chiral ring of the theory. Ignoring the Lagrange multipliers, we have

$$0 = \frac{1}{2}(YX^k + X^kY) + Y^3 = Y^3 \quad (4.11)$$

so the meson composites $(M_{jl}) = QX^jY^lQ$ are independent operators only for $j < k$ and $l < 3$. The transformation properties of the M_{jl} composites under the flavor symmetry may be determined by considering the rs element of the $N_f \times N_f$ matrix M_{jl} .

$$(M_{jl})^{rs} = Q^r X^j Y^l Q^s = (-1)^{jl} Q^r Y^l X^j Q^s \quad (4.12)$$

where we have used the vacuum condition (4.10) to anticommute Y through X , picking up a minus sign each time we do so. Now, we can use the fact that the transpose of a gauge singlet is itself to show that

$$\begin{aligned} (-1)^{jl} Q^r Y^l X^j Q^s &= (-1)^{jl} (Q^r Y^l X^j Q^s)^T = (-1)^{jl} Q^s (X^T)^j (Y^T)^l Q^r \\ &= (-1)^{jl} Q^s X^j Y^l Q^r = (-1)^{jl} (M_{jl})^{sr} \end{aligned} \quad (4.13)$$

where we have used the fact that Y and X are symmetric. We see that for jl even the meson composite M_{jl} is in the symmetric representation whereas for jl odd the meson is in the anti-symmetric representation of $SU(N_f)$.

The magnetic dual of this theory is an $SO(3kN_f + 8k + 4 - N_c)$ gauge group with charged matter

	$SO(N_c)$	$SU(N_f)$	$U(1)_R$
q	$\tilde{\mathbf{N}}_c$	$\overline{\mathbf{N}}_f$	$1 - \frac{\tilde{N}_c - 4k - 2}{N_f(k+1)}$
\bar{X}	$\frac{\tilde{\mathbf{N}}_c(\tilde{\mathbf{N}}_c+1)}{2} - \mathbf{1}$	$\mathbf{1}$	$\frac{2}{k+1}$
\bar{Y}	$\frac{\tilde{\mathbf{N}}_c(\tilde{\mathbf{N}}_c+1)}{2} - \mathbf{1}$	$\mathbf{1}$	$\frac{k}{k+1}$

and with gauge singlets $(M_{jl})^{rs}$ which are the images of the composites $Q^r X^j Y^l Q^s$ in the electric theory. The superpotential is

$$W = \frac{\text{Tr} \bar{X}^{k+1}}{k+1} + \text{Tr} \bar{X} \bar{Y}^2 + \sum_{j=0}^{k-1} \sum_{l=0}^2 M_{jl} q \bar{X}^{k-j-1} \bar{Y}^{2-l} q + \lambda_1 \text{Tr} \bar{X} + \lambda_2 \text{Tr} \bar{Y}. \quad (4.14)$$

As evidence for this claim, we cite the following.

1. The flavor symmetries of the two theories have the same global anomalies.
2. Large classes of non-redundant chiral operators appear in both theories.¹
3. For $k = 1$, the fields X and \bar{X} are massive; when integrated out they leave quartic superpotentials for Y and \bar{Y} . These theories are then dual under a duality already known from Ref. [6].
4. These theories have flat directions, analogous to those of Eq. (4.7), under which they flow to a known duality. As one moves out along these flat direction the electric theory breaks to $SO(n)^2 \times SO(m)^k$. The symmetric tensor field X contributes a field F charged as (\mathbf{n}, \mathbf{n}) under the $SO(n) \times SO(n)$ group, with superpotential $W = \text{Tr} F^{k+1}$. The dual theory breaks to $SO([k+1](N_f+2) - N_f - n)^2 \times SO(N_f + 4 - m)^k$ with similar matter content. These first two factors are an A_k loge model, related as described in Sec. 2.10 of [8], while the latter k factors are dual as discussed in [1,2].

The most significant difference between this model and the $SU(N_c)$ model of the previous section is in the flavor representations of the mesons. As noted in (4.12) and

¹ As in Refs. [6,7,8] there are unresolved issues involving certain baryon operators for SO groups.

(4.13), the mesons of this model are in the symmetric and anti-symmetric representations of the $SU(N_f)$ flavor group. The mesons M_{j1} , in particular, are symmetric for even j and anti-symmetric for odd j , and since k is odd there are $\frac{k+1}{2}$ symmetric M_{j1} and $\frac{k-1}{2}$ anti-symmetric M_{j1} . It is therefore very important for the anomaly matching that k is odd. If one naively attempts to continue k to even values, one finds that the global anomalies cannot be made to match, because the number of symmetric and anti-symmetric M_{j1} would now be equal. This is unlike the $SU(N_c)$ D_{k+2} model, where all mesons are in the $(\mathbf{N}_f, \bar{\mathbf{N}}_f)$ representation of its $SU(N_f) \times SU(N_f)$ flavor symmetry, allowing a natural anomaly matching for even k [14]. Is this naive matching a mere accident? It is certainly disturbing, given that the chiral ring does not truncate classically for even k , and there is no clear reason to restrict $j < k$ and $l < 3$. This issue remains unresolved at the time of writing.

4.3. Another D_{k+2} Orchestra Model: $Sp(N_c)$ with Two Anti-symmetric Tensors

We consider a theory of $Sp(N_c)$ gauge group with matter content

	$Sp(N_c)$	$SU(N_f)$	$U(1)_R$
Q	$\mathbf{2N}_c$	$\mathbf{2N}_f$	$1 - \frac{2N_c + 4k + 2}{2N_f(k+1)}$
X	$\frac{\mathbf{2N}_c(\mathbf{2N}_c - 1)}{2} - \mathbf{1}$	$\mathbf{1}$	$\frac{2}{k+1}$
Y	$\frac{\mathbf{2N}_c(\mathbf{2N}_c - 1)}{2} - \mathbf{1}$	$\mathbf{1}$	$\frac{k}{k+1}$

The superpotential for the theory is of the D_{k+2} form (4.9) where k is odd. The operators $Q^r X^j Y^l Q^s$ satisfy

$$Q^r X^j Y^l Q^s = (-1)^{jl} Q^r Y^l X^j Q^s = (-1)^{jl} (Q^r Y^l X^j Q^s)^T = (-1)^{jl+1} Q^s X^j Y^l Q^r \quad (4.15)$$

and so are in symmetric (anti-symmetric) representations of the flavor symmetry for jl odd (even).

The magnetic dual of this theory is an $Sp(3kN_f - 4k - 2 - N_c)$ gauge group with charged matter

	$Sp(\tilde{N}_c)$	$SU(N_f)$	$U(1)_R$
q	$\mathbf{2\tilde{N}}_c$	$\mathbf{2\bar{N}}_f$	$1 - \frac{2\tilde{N}_c + 4k + 2}{2N_f(k+1)}$
\bar{X}	$\frac{\mathbf{2\tilde{N}}_c(\mathbf{2\tilde{N}}_c - 1)}{2} - \mathbf{1}$	$\mathbf{1}$	$\frac{2}{k+1}$
\bar{Y}	$\frac{\mathbf{2\tilde{N}}_c(\mathbf{2\tilde{N}}_c - 1)}{2} - \mathbf{1}$	$\mathbf{1}$	$\frac{k}{k+1}$

and with gauge singlets $(M_{jl})^{rs}$ which are the images of the composites $Q^r X^j Y^l Q^s$ in the electric theory. The superpotential has the same form as (4.14). As in the previous cases, the usual consistency checks on the duality, such as the 't Hooft anomaly matching conditions, non-redundant operators, and the correspondence of $k = 1$ with known dualities, are satisfied.

When N_c is of the form $2n + km$ where n and m are integers, there exist flat directions in the theory analogous to the ones discussed in Sec. 4.1. As one moves out along these flat direction the electric theory breaks to $Sp(n)^2 \times Sp(m)^k$. The anti-symmetric tensor field X contributes a field F charged as (\mathbf{n}, \mathbf{n}) under the $Sp(n) \times Sp(n)$ group, with superpotential $W = \text{Tr}F^{k+1}$. The dual theory breaks to $Sp((k+1)(N_f - 1) - N_f - n)^2 \times Sp(N_f - 2 - m)^k$ with similar matter content. The $Sp(n) \times Sp(n)$ factor has the A_k loge duality discussed in Sec. 2.9 of [8], while the duality discussed in [1,3] holds in the $Sp(m)$ factors.

5. The Stage Models

5.1. $SU(N_c) \times SU(N'_c)$

This theory has a gauge group $SU(N_c) \times SU(N'_c)$ with matter content

	$SU(N_c)$	$SU(N'_c)$	$SU(N_f)_L$	$SU(N'_f)_L$	$SU(N_f)_R$	$SU(N'_f)_R$	$U(1)_R$
$Q; \tilde{Q}$	$\mathbf{N}_c; \overline{\mathbf{N}_c}$	$\mathbf{1}; \mathbf{1}$	$\mathbf{N}_f; \mathbf{1}$	$\mathbf{1}; \mathbf{1}$	$\mathbf{1}; \mathbf{N}_f$	$\mathbf{1}; \mathbf{1}$	$1 + \frac{N'_c - 2N_c}{N_f(k+1)}$
$Q'; \tilde{Q}'$	$\mathbf{1}; \mathbf{1}$	$\mathbf{N}'_c; \overline{\mathbf{N}'_c}$	$\mathbf{1}; \mathbf{1}$	$\mathbf{N}'_f; \mathbf{1}$	$\mathbf{1}; \mathbf{1}$	$\mathbf{1}; \mathbf{N}'_f$	$1 + \frac{N_c - 2N'_c}{N'_f(k+1)}$
$F; \tilde{F}$	$\mathbf{N}_c; \overline{\mathbf{N}_c}$	$\mathbf{N}'_c; \overline{\mathbf{N}'_c}$	$\mathbf{1}; \mathbf{1}$	$\mathbf{1}; \mathbf{1}$	$\mathbf{1}; \mathbf{1}$	$\mathbf{1}; \mathbf{1}$	$\frac{k}{k+1}$
X_1	$\mathbf{N}_c^2 - \mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\frac{2}{k+1}$
X_2	$\mathbf{1}$	$\mathbf{N}'_c^2 - \mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\frac{2}{k+1}$

The superpotential is

$$W = \text{Tr}X_1^{k+1} + \text{Tr}X_2^{k+1} + \text{Tr}X_1\tilde{F}F - \text{Tr}X_2\tilde{F}F + \lambda_1 \text{Tr}X_1 + \lambda_2 \text{Tr}X_2. \quad (5.1)$$

Here k can be any positive integer. It follows from the conditions for a supersymmetric vacuum that the chiral ring truncates. The gauge invariant mesons in the theory are $QX_1^j\tilde{Q}$, $Q'X_2^j\tilde{Q}'$, $QX_1^j\tilde{F}Q'$, $\tilde{Q}FX_2^j\tilde{Q}'$, $Q\tilde{F}FX_1^j\tilde{Q}$, $Q'\tilde{F}FX_2^j\tilde{Q}'$, where $j = 0 \dots k-1$.

The dual theory is described by an $SU(2kN'_f + kN_f - N'_c) \times SU(2kN_f + kN'_f - N_c)$ gauge theory with matter content

	$SU(\tilde{N}_c)$	$SU(\tilde{N}'_c)$	$SU(N_f)_L$	$SU(N'_f)_L$	$SU(N_f)_R$	$SU(N'_f)_R$	$U(1)_R$
$q; \tilde{q}$	$\tilde{\mathbf{N}}_{\mathbf{c}}; \overline{\tilde{\mathbf{N}}_{\mathbf{c}}}$	$\mathbf{1}; \mathbf{1}$	$\mathbf{1}; \mathbf{1}$	$\overline{\mathbf{N}'_{\mathbf{f}}}; \mathbf{1}$	$\mathbf{1}; \mathbf{1}$	$\mathbf{1}; \overline{\mathbf{N}'_{\mathbf{f}}}$	$1 + \frac{\tilde{N}'_c - 2\tilde{N}_c}{N'_f(k+1)}$
$q'; \tilde{q}'$	$\mathbf{1}; \mathbf{1}$	$\tilde{\mathbf{N}}'_{\mathbf{c}}; \overline{\tilde{\mathbf{N}}'_{\mathbf{c}}}$	$\overline{\mathbf{N}_{\mathbf{f}}}; \mathbf{1}$	$\mathbf{1}; \mathbf{1}$	$\mathbf{1}; \overline{\mathbf{N}_{\mathbf{f}}}$	$\mathbf{1}; \mathbf{1}$	$1 + \frac{\tilde{N}_c - 2\tilde{N}'_c}{N_f(k+1)}$
$\bar{F}; \tilde{\bar{F}}$	$\tilde{\mathbf{N}}_{\mathbf{c}}; \overline{\tilde{\mathbf{N}}_{\mathbf{c}}}$	$\tilde{\mathbf{N}}'_{\mathbf{c}}; \overline{\tilde{\mathbf{N}}'_{\mathbf{c}}}$	$\mathbf{1}; \mathbf{1}$	$\mathbf{1}; \mathbf{1}$	$\mathbf{1}; \mathbf{1}$	$\mathbf{1}; \mathbf{1}$	$\frac{k}{k+1}$
\bar{X}_1	$\tilde{\mathbf{N}}_{\mathbf{c}}^2 - \mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\frac{2}{k+1}$
\bar{X}_2	$\mathbf{1}$	$(\tilde{\mathbf{N}}'_{\mathbf{c}})^2 - \mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\frac{2}{k+1}$

There are also gauge singlet mesons in the magnetic theory which are the images of mesons in the electric theory [14]. The dual superpotential is analogous to (5.1) with the addition of coupling terms between singlets and dual mesons.

5.2. RG Flow from D_{k+2} Orchestra Model to Stage Model

In this section, we will discuss a deformation of the D_{k+2} model of Sec. 4.1 under which it flows to the stage model discussed above. In [14] the construction of the stage model was motivated by looking at the D_{k+2} model with k even, for which the duality is not clear. Here, we will see that we can in fact rigorously derive it from the well-understood duality of the k -odd case.

Consider the superpotential deformed by even powered operators

$$W = \sum_{r=1}^{\frac{k+1}{2}} \frac{s_r}{2r} \text{Tr} X^{2r} + \text{Tr} XY^2 + \lambda_1 \text{Tr} X + \lambda_2 \text{Tr} Y \quad (5.2)$$

where, as usual, we have taken k to be odd. The conditions for a supersymmetric minimum are

$$\sum_{r=1}^{\frac{k-1}{2}} s_r X^{2r} + Y^2 - \lambda_1 = 0 \quad (5.3)$$

$$XY + YX - \lambda_2 = 0.$$

We can consider tuning the couplings s_r such these become

$$\begin{aligned} X(X-a)^{\frac{k-1}{2}}(X+a)^{\frac{k-1}{2}} + Y^2 - \lambda_1 &= 0 \\ XY + YX - \lambda_2 &= 0. \end{aligned} \quad (5.4)$$

One solution to these equations and $\frac{dW}{d\lambda_1} = 0$ has $\langle \lambda_1 \rangle = \langle Y \rangle = 0$ and

$$\langle X \rangle = \begin{pmatrix} a_n & 0 & 0 \\ 0 & -a_n & 0 \\ 0 & 0 & 0_m \end{pmatrix} \quad (5.5)$$

where a subscript n indicates that an element is proportional to the $n \times n$ dimensional unit matrix. The gauge group breaks from $SU(N_c)$ to $SU(n) \times SU(n) \times SU(m) \times U(1)^2$. The off-diagonal components of X are eaten in the Higgs mechanism leaving adjoint fields X_1 , X_2 , and X_3 charged under their respective gauge groups, of which the last is massive. The field Y breaks into massless fields F and \tilde{F} charged under $SU(n) \times SU(n)$, a massless adjoint field Y_3 charged under $SU(m)$, and some massive components. The low-energy superpotential is

$$W = \text{Tr}X_1^{\frac{k+1}{2}} + \text{Tr}X_2^{\frac{k+1}{2}} + \text{Tr}X_1\tilde{F}F - \text{Tr}X_2\tilde{F}F - \frac{\text{Tr}Y_3^4}{2s_1} + \frac{(\text{Tr}Y_3^2)^2}{2s_1m}. \quad (5.6)$$

The dual gauge group breaks similarly from $SU(3kN_f - N_c)$ to $SU(\frac{3(k-1)}{2}N_f - n) \times SU(\frac{3(k-1)}{2}N_f - n) \times SU(3N_f - m) \times U(1)^2$. The stage model duality relates the $SU(n) \times SU(n)$ factor to the $SU(\frac{3(k-1)}{2}N_f - n) \times SU(\frac{3(k-1)}{2}N_f - n)$ factor, while the $SU(m)$ theory and its dual are of A_3 type [4].

5.3. A New Stage Model: $SO(N_c) \times SO(N'_c)$

The electric $SO(N_c) \times SO(N'_c)$ theory has matter content

	$SO(N_c)$	$SO(N'_c)$	$SU(N_f)$	$SU(N'_f)$	$U(1)_R$
Q	\mathbf{N}_c	$\mathbf{1}$	\mathbf{N}_f	$\mathbf{1}$	$1 + \frac{N'_c - 2N_c + 4k}{N_f(k+1)}$
Q'	$\mathbf{1}$	\mathbf{N}'_c	$\mathbf{1}$	\mathbf{N}'_f	$1 + \frac{N_c - 2N'_c + 4k}{N'_f(k+1)}$
F	\mathbf{N}_c	\mathbf{N}'_c	$\mathbf{1}$	$\mathbf{1}$	$\frac{k}{k+1}$
X_1	$\frac{\mathbf{N}_c(\mathbf{N}_c+1)}{2} - \mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\frac{2}{k+1}$
X_2	$\mathbf{1}$	$\frac{\mathbf{N}'_c(\mathbf{N}'_c+1)}{2} - \mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\frac{2}{k+1}$

The electric superpotential is

$$W = \text{Tr}X_1^{k+1} + \text{Tr}X_2^{k+1} + \text{Tr}X_1F^2 - \text{Tr}X_2F^2 + \lambda_1 \text{Tr}X_1 + \lambda_2 \text{Tr}X_2. \quad (5.7)$$

This superpotential truncates the chiral ring for all values of k . The mesons $M_j = QX_1^jQ$, $M'_j = Q'X_2^jQ'$, $N_j = QX_1^jF^2Q$, and $N'_j = Q'X_2^jF^2Q'$ are symmetric under the flavor groups, while the mesons $P_j = QX_1^jFQ'$ are in the $(\mathbf{N}_f, \mathbf{N}'_f)$ representation.

The magnetic theory is of similar type, with gauge group $SO(2kN'_f + kN_f + 8k - N'_c) \times SO(2kN_f + kN'_f + 8k - N_c)$ and matter content

	$SO(\tilde{N}_c)$	$SO(\tilde{N}'_c)$	$SU(N_f)$	$SU(N'_f)$	$U(1)_R$
q	$\tilde{\mathbf{N}}_{\mathbf{c}}$	$\mathbf{1}$	$\mathbf{1}$	$\overline{\mathbf{N}'_{\mathbf{f}}}$	$1 + \frac{\tilde{N}'_c - 2\tilde{N}_c + 4k}{N'_f(k+1)}$
q'	$\mathbf{1}$	$\tilde{\mathbf{N}}'_{\mathbf{c}}$	$\overline{\mathbf{N}'_{\mathbf{f}}}$	$\mathbf{1}$	$1 + \frac{\tilde{N}_c - 2\tilde{N}'_c + 4k}{N_f(k+1)}$
\bar{F}	$\tilde{\mathbf{N}}_{\mathbf{c}}$	$\tilde{\mathbf{N}}'_{\mathbf{c}}$	$\mathbf{1}$	$\mathbf{1}$	$\frac{k}{k+1}$
\bar{X}_1	$\frac{\tilde{\mathbf{N}}_{\mathbf{c}}(\tilde{\mathbf{N}}_{\mathbf{c}}+1)}{2} - \mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\frac{2}{k+1}$
\bar{X}_2	$\mathbf{1}$	$\frac{(\tilde{\mathbf{N}}'_{\mathbf{c}}(\tilde{\mathbf{N}}'_{\mathbf{c}}+1)}{2} - \mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\frac{2}{k+1}$

The mesons M, M', N, N', P of the electric theory are mapped to singlets of the dual theory in the usual way. The dual superpotential has the form

$$\begin{aligned}
W = & \text{Tr}\bar{X}_1^{k+1} + \text{Tr}\bar{X}_2^{k+1} + \text{Tr}\bar{X}_1\bar{F}^2 - \text{Tr}\bar{X}_2\bar{F}^2 + \lambda_1 \text{Tr}\bar{X}_1 + \lambda_2 \text{Tr}\bar{X}_2 \\
& + \sum_{j=0}^{k-1} \left\{ M_j q' \bar{X}_2^{k-j-1} \bar{F}^2 q' + M'_j q \bar{X}_1^{k-j-1} \bar{F}^2 q \right. \\
& \quad \left. + P_j q \bar{X}_1^{k-j-1} \bar{F} q' + N_j q' \bar{X}_2^{k-j-1} q' + N'_j q \bar{X}_1^{k-j-1} q \right\}.
\end{aligned} \tag{5.8}$$

5.4. RG Flow from D_{k+2} Orchestra Model to Stage Model

In analogy to Sec. 5.2, deforming the superpotential of the $SO(N_c)$ gauge theory discussed in Sec. 4.2 by even powers $\text{Tr}X^{2r}$ can cause it to flow from the D_{k+2} orchestra theory to this stage model. The superpotential has the form

$$W = \sum_{r=1}^{\frac{k+1}{2}} \frac{s_r}{2r} \text{Tr}X^{2r} + \text{Tr}XY^2 - \lambda_1 \text{Tr}X - \lambda_2 \text{Tr}Y \tag{5.9}$$

where, as usual, we have taken k to be odd. As described in Sec. 5.2, by tuning the couplings s_r we can find a superpotential which has a vacuum $\langle Y \rangle = 0$ and

$$\langle X \rangle = \begin{pmatrix} a_n & 0 & 0 \\ 0 & -a_n & 0 \\ 0 & 0 & 0_m \end{pmatrix} \tag{5.10}$$

The electric theory breaks from $SO(N_c)$ down to $SO(n) \times SO(n) \times SO(m)$. The low energy superpotential is

$$W = \text{Tr}X_1^{\frac{k+1}{2}} + \text{Tr}X_1F^2 + \text{Tr}X_2^{\frac{k+1}{2}} - \text{Tr}X_2F^2 - \frac{\text{Tr}Y_3^4}{2s_1} + \frac{(\text{Tr}Y_3^2)^2}{2s_1m}. \tag{5.11}$$

where the fields X_1, X_2 , are symmetric tensors coming from X , Y_3 is a symmetric tensor and F is a field in the (\mathbf{n}, \mathbf{n}) representation coming from Y . The dual gauge group breaks from $SO(3kN_f + 8k + 4 - N_c)$ to $SO(3\frac{(k-1)}{2}N_f + 4k - 4 - n) \times SO(3\frac{(k-1)}{2}N_f + 4k - 4 - n) \times SO(3N_f + 12 - m)$, which is dual to the low-energy electric theory under the stage and A_3 [6] duality transformations.

5.5. Another New Stage Model: $Sp(N_c) \times Sp(N'_c)$

Consider the gauge theory $Sp(N_c) \times Sp(N'_c)$ with matter content

	$Sp(N_c)$	$Sp(N'_c)$	$SU(N_f)$	$SU(N'_f)$	$U(1)_R$
Q	$\mathbf{2N}_c$	$\mathbf{1}$	$\mathbf{2N}_f$	$\mathbf{1}$	$1 + \frac{N'_c - 2N_c - 2k}{N_f(k+1)}$
Q'	$\mathbf{1}$	$\mathbf{2N}'_c$	$\mathbf{1}$	$\mathbf{2N}'_f$	$1 + \frac{N_c - 2N'_c - 2k}{N'_f(k+1)}$
F	$\mathbf{2N}_c$	$\mathbf{2N}'_c$	$\mathbf{1}$	$\mathbf{1}$	$\frac{k}{k+1}$
X_1	$\frac{\mathbf{2N}_c(\mathbf{2N}_c - 1)}{2} - \mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\frac{2}{k+1}$
X_2	$\mathbf{1}$	$\frac{\mathbf{2N}'_c(\mathbf{2N}'_c - 1)}{2} - \mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\frac{2}{k+1}$

The electric superpotential takes the form (5.7) and truncates the chiral ring for all values of k . The mesons in the theory anti-symmetric under the flavor groups are $QX_1^j Q$, $Q'X_2^j Q'$, $QX_1^j F^2 Q$, and $Q'X_2^j F^2 Q'$. The mesons in the theory in the $(\mathbf{2N}_f, \mathbf{2N}'_f)$ representation are $QX_1^j F Q'$.

The magnetic theory is of similar type, with gauge group $Sp(2kN'_f + kN_f - 4k - N'_c) \times Sp(2kN_f + kN'_f - 4k - N_c)$ and matter content

	$Sp(\tilde{N}_c)$	$Sp(\tilde{N}'_c)$	$SU(N_f)$	$SU(N'_f)$	$U(1)_R$
q	$\mathbf{2}\tilde{\mathbf{N}}_c$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}\overline{\mathbf{N}}'_f$	$1 + \frac{\tilde{N}'_c - 2\tilde{N}_c - 2k}{N'_f(k+1)}$
q'	$\mathbf{1}$	$\mathbf{2}\tilde{\mathbf{N}}'_c$	$\mathbf{2}\overline{\mathbf{N}}_f$	$\mathbf{1}$	$1 + \frac{\tilde{N}_c - 2\tilde{N}'_c - 2k}{N_f(k+1)}$
\bar{F}	$\mathbf{2}\tilde{\mathbf{N}}_c$	$\mathbf{2}\tilde{\mathbf{N}}'_c$	$\mathbf{1}$	$\mathbf{1}$	$\frac{k}{k+1}$
\bar{X}_1	$\frac{\mathbf{2}\tilde{\mathbf{N}}_c(\mathbf{2}\tilde{\mathbf{N}}_c - 1)}{2} - \mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\frac{2}{k+1}$
\bar{X}_2	$\mathbf{1}$	$\frac{\mathbf{2}\tilde{\mathbf{N}}'_c(\mathbf{2}\tilde{\mathbf{N}}'_c - 1)}{2} - \mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\frac{2}{k+1}$

The mesons of the electric theory are mapped to singlets of the dual theory in the usual way.

In analogy to Sec. 5.2, deforming the $Sp(N_c)$ D_{k+2} superpotential by even powers $\text{Tr}X^{2r}$ can cause the $Sp(N_c)$ orchestra theory of Sec. 4.3 to flow to the stage model of this section. When the electric theory breaks to $Sp(n) \times Sp(n) \times Sp(m)$, the dual gauge group breaks from $Sp(3kN_f - 4k - 2 - N_c)$ to $Sp(3\frac{(k-1)}{2}N_f - 2k + 2 - n) \times Sp(3\frac{(k-1)}{2}N_f - 2k + 2 - n) \times Sp(3N_f - 6 - m)$. The $Sp(n) \times Sp(n)$ factors are dual as described above, and the A_3 duality of [6] applies in the $Sp(m)$ sector.

6. D_{k+2} Mezzanine Dualities

6.1. Mezzanine: $SO(N_c)$ with a Symmetric and an Anti-symmetric Tensor

We consider an $SO(N_c)$ gauge theory with

	$SO(N_c)$	$SU(N_f)$	$U(1)_R$
Q	\mathbf{N}_c	\mathbf{N}_f	$1 - \frac{N_c - 4k + 2}{N_f(k+1)}$
X	$\frac{\mathbf{N}_c(\mathbf{N}_c+1)}{2} - \mathbf{1}$	$\mathbf{1}$	$\frac{2}{k+1}$
Y	$\frac{\mathbf{N}_c(\mathbf{N}_c-1)}{2}$	$\mathbf{1}$	$\frac{k}{k+1}$

The superpotential for the theory is of the D_{k+2} form

$$W = \frac{\text{Tr} X^{k+1}}{k+1} + \text{Tr} XY^2 + \lambda \text{Tr} X . \quad (6.1)$$

k is odd. The meson composites QX^jY^lQ are independent operators only for $j < k$ and $l < 3$. Furthermore, the operators $Q^rX^jY^lQ^s$ satisfy

$$Q^rX^jY^lQ^s = (-1)^{jl}Q^rY^lX^jQ^s = (-1)^{jl}(Q^rY^lX^jQ^s)^T = (-1)^{(j+1)l}Q^sX^jY^lQ^r \quad (6.2)$$

and so are in symmetric (anti-symmetric) representations of the flavor symmetry for $(j+1)l$ even (odd).

The magnetic dual of this theory is an $SO(3kN_f + 8k - 4 - N_c)$ gauge group with charged fields

	$SO(\tilde{N}_c)$	$SU(N_f)$	$U(1)_R$
q	$\tilde{\mathbf{N}}_c$	$\overline{\mathbf{N}}_f$	$1 - \frac{\tilde{N}_c - 4k + 2}{N_f(k+1)}$
\bar{X}	$\frac{\tilde{\mathbf{N}}_c(\tilde{\mathbf{N}}_c+1)}{2} - \mathbf{1}$	$\mathbf{1}$	$\frac{2}{k+1}$
\bar{Y}	$\frac{\tilde{\mathbf{N}}_c(\tilde{\mathbf{N}}_c-1)}{2}$	$\mathbf{1}$	$\frac{k}{k+1}$

and with gauge singlets $(M_{jl})^{rs}$ which are the images of the composites $Q^rX^jY^lQ^s$ in the electric theory. The dual superpotential is of the form

$$W = \frac{\text{Tr} \bar{X}^{k+1}}{k+1} + \text{Tr} \bar{X} \bar{Y}^2 + \sum_{j=0}^{k-1} \sum_{l=0}^2 M_{jl} q \bar{X}^{k-j-1} \bar{Y}^{2-l} q + \lambda \text{Tr} \bar{X} . \quad (6.3)$$

As in the previous cases, we find that the usual consistency checks on the duality are satisfied.

In analogy to Sec. 5.2, deformations of the superpotential by even powers of $\text{Tr}X^{2r}$ can cause the theory to flow to the stage model discussed in Sec. 5.3. When the electric theory breaks to $SO(n) \times SO(n) \times SO(m)$, the dual gauge group breaks from $SO(3kN_f + 8k - 4 - N_c)$ to $SO(3\frac{(k-1)}{2}N_f + 4k - 4 - n) \times SO(3\frac{(k-1)}{2}N_f + 4k - 4 - n) \times SO(3N_f + 4 - m)$. The duality of Sec. 5.3 applies in the $SO(n) \times SO(n)$ sector and the duality of [7] applies in the $SO(m)$ sector.

6.2. Mezzanine: $Sp(N_c)$ with an Anti-symmetric and a Symmetric Tensor

We consider a theory with $Sp(N_c)$ gauge group and matter content

	$Sp(N_c)$	$SU(N_f)$	$U(1)_R$
Q	$\mathbf{2N_c}$	$\mathbf{2N_f}$	$1 - \frac{N_c+2k-1}{N_f(k+1)}$
X	$\frac{\mathbf{2N_c}(2N_c-1)}{2} - \mathbf{1}$	$\mathbf{1}$	$\frac{2}{k+1}$
Y	$\frac{\mathbf{2N_c}(2N_c+1)}{2}$	$\mathbf{1}$	$\frac{k}{k+1}$

The superpotential for the theory is of the D_{k+2} form (6.1) where k is odd. Furthermore, the operators $Q^r X^j Y^l Q^s$ satisfy

$$Q^r X^j Y^l Q^s = (-1)^{jl} Q^r Y^l X^j Q^s = (-1)^{jl} (Q^r Y^l X^j Q^s)^T = (-1)^{j+l+1} Q^s X^j Y^l Q^r \quad (6.4)$$

and so are in symmetric (anti-symmetric) representations of the flavor symmetry for $(j+1)l$ odd (even).

The magnetic dual of this theory is an $Sp(3kN_f - 4k + 2 - N_c)$ gauge group with charged matter

	$Sp(\tilde{N}_c)$	$SU(N_f)$	$U(1)_R$
q	$\mathbf{2\tilde{N}_c}$	$\mathbf{2\overline{N}_f}$	$1 - \frac{\tilde{N}_c+2k-1}{N_f(k+1)}$
\bar{X}	$\frac{\mathbf{2\tilde{N}_c}(2\tilde{N}_c-1)}{2} - \mathbf{1}$	$\mathbf{1}$	$\frac{2}{k+1}$
\bar{Y}	$\frac{\mathbf{2\tilde{N}_c}(2\tilde{N}_c+1)}{2}$	$\mathbf{1}$	$\frac{k}{k+1}$

and with gauge singlets $(M_{jl})^{rs}$ which are the images of the composites $Q^r X^j Y^l Q^s$ in the electric theory. The superpotential has the form (6.3). All the usual consistency checks are satisfied.

As in Sec. 5.2, by deforming the superpotential by even powers $\text{Tr}X^{2r}$, one can flow from this mezzanine theory to the stage model discussed in Sec. 5.5. When the electric

theory breaks to $Sp(n) \times Sp(n) \times Sp(m)$, the dual gauge group breaks from $Sp(3kN_f - 4k + 2 - N_c)$ to $Sp(3\frac{(k-1)}{2}N_f - 2k + 2 - n) \times Sp(3\frac{(k-1)}{2}N_f - 2k + 2 - n) \times Sp(3N_f - 2 - m)$. The duality in the $Sp(n) \times Sp(n)$ sector is that of Sec. 5.5 and the duality in the $Sp(m)$ sector is that of [7].

6.3. Mezzanine: $SU(N_c)$ with Adjoint, Symmetric, and Conjugate Symmetric Tensors

The $SU(N_c)$ gauge theory has the following matter content:

	$SU(N_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_Y$	$U(1)_B$	$U(1)_R$
Q	\mathbf{N}_c	\mathbf{N}_f	$\mathbf{1}$	0	$\frac{1}{N_c}$	$1 - \frac{N_c - 2}{N_f(k+1)}$
\tilde{Q}	$\overline{\mathbf{N}_c}$	$\mathbf{1}$	$\overline{\mathbf{N}_f}$	0	$-\frac{1}{N_c}$	$1 - \frac{N_c - 2}{N_f(k+1)}$
X	$\mathbf{N}_c^2 - \mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	$\frac{2}{k+1}$
Y	$\frac{\mathbf{N}_c(\mathbf{N}_c+1)}{2}$	$\mathbf{1}$	$\mathbf{1}$	1	$\frac{2}{N_c}$	$\frac{k}{k+1}$
\tilde{Y}	$\frac{\mathbf{N}_c(\mathbf{N}_c+1)}{2}$	$\mathbf{1}$	$\mathbf{1}$	-1	$-\frac{2}{N_c}$	$\frac{k}{k+1}$

The superpotential is

$$W = \frac{\text{Tr}X^{k+1}}{k+1} + \text{Tr}XY\tilde{Y} + \lambda \text{Tr}X \quad (6.5)$$

where λ is a Lagrange multiplier. The conditions for a supersymmetric minimum that follow from this superpotential are

$$\begin{aligned} X^k + Y\tilde{Y} + \lambda &= 0 \\ \frac{XY + YX^T}{2} &= 0 \\ \frac{X\tilde{Y} + \tilde{Y}X^T}{2} &= 0. \end{aligned} \quad (6.6)$$

These equations truncate the chiral ring for odd values of k . As usual, we ignore λ . We can multiply X^k by Y from the right to form $X^kY = -Y\tilde{Y}Y$. We can then take the transpose of the first of (6.6) and multiply it from the left by Y . Combining these two equations we have

$$Y(X^T)^k + X^kY = -2Y\tilde{Y}Y. \quad (6.7)$$

Now, we can use the second equation in (6.6) to show that

$$((-1)^k + 1)X^kY = -2Y\tilde{Y}Y. \quad (6.8)$$

Thus for odd k , $Y\tilde{Y}Y = 0$. Likewise, $\tilde{Y}YY\tilde{Y} = 0$. In this theory, one can construct gauge invariant mesons in the $(\mathbf{N}_f, \overline{\mathbf{N}}_f)$ representation

$$\begin{aligned}(N_j)_s^r &= Q^r X^j \tilde{Y} Y \tilde{Q}_s \\ (M_j)_s^r &= Q^r X^j \tilde{Q}_s\end{aligned}\tag{6.9}$$

where $j = 0, 1, \dots, k-1$, while the mesons

$$\begin{aligned}(P_j)^{rs} &= Q^r X^j \tilde{Y} Q^s = (-1)^j Q^r \tilde{Y} (X^T)^j Q^s \\ &= (-1)^j (Q^r \tilde{Y} (X^T)^j Q^s)^T = (-1)^j Q^s X^j \tilde{Y} Q^r \\ (\tilde{P}_j)_{rs} &= \tilde{Q}_r Y X^j \tilde{Q}_s = (-1)^j \tilde{Q}_r (X^T)^j Y \tilde{Q}_s \\ &= (-1)^j (\tilde{Q}_r (X^T)^j Y \tilde{Q}_s)^T = (-1)^j \tilde{Q}_s Y X^j \tilde{Q}_r\end{aligned}\tag{6.10}$$

are in the (anti)-symmetric representation of the flavor group for (odd) even values of j where $j = 0, 1, \dots, k-1$.

This theory has a dual description in terms of an $SU(3kN_f + 4 - N_c)$ gauge theory with the following charged fields

	$SU(\tilde{N}_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_Y$	$U(1)_B$	$U(1)_R$
q	$\tilde{\mathbf{N}}_c$	$\overline{\mathbf{N}}_f$	$\mathbf{1}$	$\frac{kN_f+2}{\tilde{N}_c}$	$\frac{1}{\tilde{N}_c}$	$1 - \frac{\tilde{N}_c-2}{N_f(k+1)}$
\tilde{q}	$\overline{\tilde{\mathbf{N}}_c}$	$\mathbf{1}$	\mathbf{N}_f	$-\frac{kN_f+2}{\tilde{N}_c}$	$-\frac{1}{\tilde{N}_c}$	$1 - \frac{\tilde{N}_c-2}{N_f(k+1)}$
\bar{X}	$\tilde{\mathbf{N}}_c^2 - \mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	$\frac{2}{k+1}$
\bar{Y}	$\frac{\tilde{\mathbf{N}}_c(\tilde{\mathbf{N}}_c+1)}{2}$	$\mathbf{1}$	$\mathbf{1}$	$\frac{N_c-kN_f}{\tilde{N}_c}$	$\frac{2}{\tilde{N}_c}$	$\frac{k}{k+1}$
$\tilde{\bar{Y}}$	$\frac{\tilde{\mathbf{N}}_c(\tilde{\mathbf{N}}_c+1)}{2}$	$\mathbf{1}$	$\mathbf{1}$	$-\frac{N_c-kN_f}{\tilde{N}_c}$	$-\frac{2}{\tilde{N}_c}$	$\frac{k}{k+1}$

The dual description also has singlets, $(N_j)_s^r, (M_j)_s^r, (P_j)^{rs}, (\tilde{P}_j)_{rs}$, which are in a one-to-one mapping with the mesons in Eqs. (6.9)-(6.10). The dual superpotential has the form

$$\begin{aligned}W &= \frac{\text{Tr } \bar{X}^{k+1}}{k+1} + \text{Tr } \bar{X} \bar{Y} \tilde{\bar{Y}} + \lambda \text{Tr } \bar{X} \\ &\quad + \sum_{j=0}^{k-1} \left\{ N_j \tilde{q} \bar{X}^{k-j-1} q + P_j q \bar{X}^{k-j-1} \bar{Y} q + \tilde{P}_j \tilde{q} \tilde{\bar{Y}} \bar{X}^{k-j-1} \tilde{q} + M_j \tilde{q} \bar{X}^{k-j-1} \tilde{\bar{Y}} \bar{Y} q \right\}.\end{aligned}\tag{6.11}$$

As in the previous cases, we find that the usual consistency checks on the duality are satisfied.

As in Sec. 5.2, by deforming the superpotential by even powers $\text{Tr}X^{2r}$, one can flow from this theory to the stage model discussed in Sec. 5.1. When the electric group breaks to $SU(n) \times SU(n) \times SU(m)$, the dual gauge group breaks from $SU(3kN_f + 4 - N_c)$ to $SU(\frac{3(k-1)}{2}N_f - n) \times SU(\frac{3(k-1)}{2}N_f - n) \times SU(3N_f + 4 - m) \times U(1)^2$. The duality in the $SU(n) \times SU(n)$ part of the theory was described in Sec. 5.1 and in [14]; that of the $SU(m)$ part was described in [8].

6.4. Mezzanine: $SU(N_c)$ with Adjoint, Anti-symmetric, and Conjugate Anti-symmetric Tensors

The $SU(N_c)$ gauge theory has the following matter content:

	$SU(N_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_Y$	$U(1)_B$	$U(1)_R$
Q	\mathbf{N}_c	\mathbf{N}_f	$\mathbf{1}$	0	$\frac{1}{N_c}$	$1 - \frac{N_c+2}{N_f(k+1)}$
\tilde{Q}	$\overline{\mathbf{N}_c}$	$\mathbf{1}$	$\overline{\mathbf{N}_f}$	0	$-\frac{1}{N_c}$	$1 - \frac{N_c+2}{N_f(k+1)}$
X	$\mathbf{N}_c^2 - \mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	$\frac{2}{k+1}$
Y	$\frac{\mathbf{N}_c(\mathbf{N}_c-1)}{2}$	$\mathbf{1}$	$\mathbf{1}$	1	$\frac{2}{N_c}$	$\frac{k}{k+1}$
\tilde{Y}	$\frac{\mathbf{N}_c(\mathbf{N}_c-1)}{2}$	$\mathbf{1}$	$\mathbf{1}$	-1	$-\frac{2}{N_c}$	$\frac{k}{k+1}$

The superpotential is of the form (6.5), and as before the chiral ring truncates for k odd. In this theory, one can construct gauge invariant mesons in the $(\mathbf{N}_f, \overline{\mathbf{N}}_f)$ representation

$$\begin{aligned} (N_j)_s^r &= Q^r X^j \tilde{Y} Y \tilde{Q}_s \\ (M_j)_s^r &= Q^r X^j \tilde{Q}_s \end{aligned} \tag{6.12}$$

where $j = 0, 1, \dots, k-1$, while the mesons

$$\begin{aligned} P_j &= Q^r X^j \tilde{Y} Q^s = (-1)^j Q^r \tilde{Y} (X^T)^j Q^s \\ &= (-1)^j (Q^r \tilde{Y} (X^T)^j Q^s)^T = (-1)^{j+1} Q^s X^j \tilde{Y} Q^r \\ \tilde{P}_j &= \tilde{Q}_r Y X^j \tilde{Q}_s = (-1)^j \tilde{Q}_r (X^T)^j Y \tilde{Q}_s \\ &= (-1)^j (\tilde{Q}_r (X^T)^j Y \tilde{Q}_s)^T = (-1)^{j+1} \tilde{Q}_s Y X^j \tilde{Q}_r \end{aligned} \tag{6.13}$$

are in the (anti)-symmetric representation of the flavor group for (even) odd values of j where $j = 0, 1, \dots, k-1$.

This theory has a dual description in terms of an $SU(3kN_f - 4 - N_c)$ gauge theory with the following charged fields:

	$SU(\tilde{N}_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_Y$	$U(1)_B$	$U(1)_R$
q	$\tilde{\mathbf{N}}_{\mathbf{c}}$	$\overline{\mathbf{N}_{\mathbf{f}}}$	$\mathbf{1}$	$\frac{kN_f - 2}{\tilde{N}_c}$	$\frac{1}{\tilde{N}_c}$	$1 - \frac{\tilde{N}_c + 2}{N_f(k+1)}$
\tilde{q}	$\overline{\tilde{\mathbf{N}}_{\mathbf{c}}}$	$\mathbf{1}$	$\mathbf{N}_{\mathbf{f}}$	$-\frac{kN_f - 2}{\tilde{N}_c}$	$-\frac{1}{\tilde{N}_c}$	$1 - \frac{\tilde{N}_c + 2}{N_f(k+1)}$
\bar{X}	$\tilde{\mathbf{N}}_{\mathbf{c}}^2 - \mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	$\frac{2}{k+1}$
\bar{Y}	$\frac{\tilde{\mathbf{N}}_{\mathbf{c}}(\tilde{\mathbf{N}}_{\mathbf{c}} - 1)}{2}$	$\mathbf{1}$	$\mathbf{1}$	$\frac{N_c - kN_f}{N_c}$	$\frac{2}{N_c}$	$\frac{k}{k+1}$
$\tilde{\bar{Y}}$	$\frac{\tilde{\mathbf{N}}_{\mathbf{c}}(\tilde{\mathbf{N}}_{\mathbf{c}} - 1)}{2}$	$\mathbf{1}$	$\mathbf{1}$	$-\frac{N_c - kN_f}{N_c}$	$-\frac{2}{N_c}$	$\frac{k}{k+1}$

The dual description also has singlets, $(N_j)_s^r$, $(M_j)_s^r$, $(P_j)^{rs}$, $(\tilde{P}_j)_{rs}$, which are in a one-to-one mapping with the mesons in Eqs. (6.12)-(6.13). The dual superpotential has the form (6.11). As in the previous cases, we find that the usual consistency checks on the duality are satisfied.

As in Sec. 5.2, by deforming the superpotential by even powers $\text{Tr}X^{2r}$, one can flow from this mezzanine theory to the stage model discussed in Sec. 5.1. When the electric group breaks to $SU(n) \times SU(n) \times SU(m)$, the dual gauge group breaks from $SU(3kN_f - 4 - N_c)$ to $SU(\frac{3(k-1)}{2}N_f - n) \times SU(\frac{3(k-1)}{2}N_f - n) \times SU(3N_f - 4 - m) \times U(1)^2$. The duality of the $SU(n) \times SU(n)$ part of the theory was described in Sec. 5.1 and in [14]; that of the $SU(m)$ part was described in Sec. 2.6 of [8].

7. D_{k+2} Balcony Model: $SU(N_c)$ with Adjoint, Anti-symmetric and Conjugate Symmetric Tensors

The $SU(N_c)$ gauge theory has the following matter content:

	$SU(N_c)$	$SU(m_f)$	$SU(\tilde{m}_f)$	$U(1)_Y$	$U(1)_B$	$U(1)_R$
Q	$\mathbf{N}_{\mathbf{c}}$	$\mathbf{m}_{\mathbf{f}}$	$\mathbf{1}$	$\frac{6}{m_f + 4} - 1$	$\frac{1}{N_c}$	$1 - \frac{N_c + 6k}{m_f(k+1)}$
\tilde{Q}	$\overline{\mathbf{N}_{\mathbf{c}}}$	$\mathbf{1}$	$\tilde{\mathbf{m}}_{\mathbf{f}}$	$\frac{6}{m_f - 4} + 1$	$-\frac{1}{N_c}$	$1 - \frac{N_c - 6k}{\tilde{m}_f(k+1)}$
X	$\mathbf{N}_{\mathbf{c}}^2 - \mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	$\frac{2}{k+1}$
Y	$\frac{\mathbf{N}_{\mathbf{c}}(\mathbf{N}_{\mathbf{c}} - 1)}{2}$	$\mathbf{1}$	$\mathbf{1}$	1	$\frac{2}{N_c}$	$\frac{k}{k+1}$
\tilde{Y}	$\frac{\mathbf{N}_{\mathbf{c}}(\mathbf{N}_{\mathbf{c}} + 1)}{2}$	$\mathbf{1}$	$\mathbf{1}$	-1	$-\frac{2}{N_c}$	$\frac{k}{k+1}$

This theory is chiral, and the anomaly cancellation requirement is that $m_f = \tilde{m}_f + 8$. The superpotential is

$$W = \frac{\text{Tr}X^{k+1}}{k+1} + \text{Tr}XY\tilde{Y} + \lambda \text{Tr}X \quad (7.1)$$

where λ is a Lagrange multiplier. It follows from the conditions for a supersymmetric minimum that the chiral ring truncates for all values of k , odd or even. This is in contrast with all the other examples in this paper, for which the chiral ring truncates only for k odd. To see why the k even case is different here, we consider the vacuum conditions that follow from this superpotential.

$$\begin{aligned} X^k + Y\tilde{Y} + \lambda &= 0 \\ \frac{XY - YX^T}{2} &= 0 \\ \frac{X\tilde{Y} - \tilde{Y}X^T}{2} &= 0. \end{aligned} \tag{7.2}$$

As usual, we ignore λ . We can multiply X^k by Y from the right to form $X^k Y = -Y\tilde{Y}Y$. We can then take the transpose of the first of (7.2) and multiply it from the left by Y to form $Y(X^T)^k = +Y\tilde{Y}Y$. Combining these two equations we have

$$Y(X^T)^k - X^k Y = 2Y\tilde{Y}Y. \tag{7.3}$$

Now, we can use the second equation in (7.2) to show that for all values of k , $Y\tilde{Y}Y = 0$. Likewise, $\tilde{Y}YY = 0$. In this theory, one can construct gauge invariant mesons in the $(\mathbf{N}_f, \overline{\mathbf{N}}_f)$ representation

$$\begin{aligned} (N_j)_s^r &= Q^r X^j \tilde{Y} Y \tilde{Q}_s \\ (M_j)_s^r &= Q^r X^j \tilde{Q}_s \end{aligned} \tag{7.4}$$

where $j = 0, 1, \dots, k-1$, while the mesons

$$\begin{aligned} P_j &= Q^r X^j \tilde{Y} Q^s = Q^r \tilde{Y} (X^T)^j Q^s \\ &= (Q^r \tilde{Y} (X^T)^j Q^s)^T = Q^s X^j \tilde{Y} Q^r \\ \tilde{P}_j &= \tilde{Q}_r Y X^j \tilde{Q}_s = \tilde{Q}_r (X^T)^j Y \tilde{Q}_s \\ &= (\tilde{Q}_r (X^T)^j Y \tilde{Q}_s)^T = (-1) \tilde{Q}_s Y X^j \tilde{Q}_r \end{aligned} \tag{7.5}$$

are in the symmetric (anti-symmetric) representation of the flavor group.

This theory has a dual description in terms of a $SU(3k \frac{(m_f + \tilde{m}_f)}{2} - N_c)$ gauge theory with the following matter fields

	$SU(\tilde{N}_c)$	$SU(m_f)$	$SU(\tilde{m}_f)$	$U(1)_Y$	$U(1)_B$	$U(1)_R$
q	$\tilde{\mathbf{N}}_{\mathbf{c}}$	$\overline{\mathbf{m}}_{\mathbf{f}}$	$\mathbf{1}$	$1 - \frac{6}{m_f+4}$	$\frac{1}{\tilde{N}_c}$	$1 - \frac{\tilde{N}_c+6k}{m_f(k+1)}$
\tilde{q}	$\overline{\tilde{\mathbf{N}}_{\mathbf{c}}}$	1	$\overline{\tilde{\mathbf{m}}_{\mathbf{f}}}$	$-1 - \frac{6}{m_f+4}$	$-\frac{1}{\tilde{N}_c}$	$1 - \frac{\tilde{N}_c-6k}{\tilde{m}_f(k+1)}$
\bar{X}	$\tilde{\mathbf{N}}_{\mathbf{c}}^2 - \mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	$\frac{2}{k+1}$
\bar{Y}	$\frac{\tilde{\mathbf{N}}_{\mathbf{c}}(\tilde{\mathbf{N}}_{\mathbf{c}}-1)}{2}$	$\mathbf{1}$	$\mathbf{1}$	1	$\frac{2}{\tilde{N}_c}$	$\frac{k}{k+1}$
$\tilde{\bar{Y}}$	$\frac{\tilde{\mathbf{N}}_{\mathbf{c}}(\tilde{\mathbf{N}}_{\mathbf{c}}+1)}{2}$	$\mathbf{1}$	$\mathbf{1}$	-1	$-\frac{2}{\tilde{N}_c}$	$\frac{k}{k+1}$

The dual description also has singlets, $(N_j)_s^r, (M_j)_s^r, (P_j)^{rs}, (\tilde{P}_j)_{rs}$, which are in a one-to-one mapping with the mesons in Eqs. (7.4)-(7.5). The dual superpotential has the form

$$W = \frac{\text{Tr} \bar{X}^{k+1}}{k+1} + \text{Tr} \bar{X} \bar{Y} \tilde{\bar{Y}} + \lambda \text{Tr} \bar{X} \\ + \sum_{j=0}^{k-1} \left\{ N_j \tilde{q} \bar{X}^{k-j-1} q + P_j q \bar{X}^{k-j-1} \bar{Y} q + \tilde{P}_j \tilde{q} \tilde{\bar{Y}} \bar{X}^{k-j-1} \tilde{q} + M_j \tilde{q} \bar{X}^{k-j-1} \tilde{\bar{Y}} q \right\}. \quad (7.6)$$

As in the previous cases, we find that the usual consistency checks on the duality are satisfied.

For $N_c = kn$, this theory has flat directions

$$\langle X \rangle = a \begin{pmatrix} 1_n & 0 & 0 & 0 & . & 0 \\ 0 & \omega_n & 0 & 0 & . & 0 \\ 0 & 0 & \omega_n^2 & 0 & . & 0 \\ 0 & 0 & 0 & \omega_n^3 & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & \omega_n^{k-1} \end{pmatrix} \quad (7.7)$$

which break the theory down to $SU(n)^k$. In each $SU(n)$ factor, the adjoint field from X is massive while anti-symmetric and conjugate symmetric tensors \hat{Y} and $\tilde{\hat{Y}}$ are massless, with low-energy superpotential $W = \text{Tr}(\hat{Y} \hat{Y})^2$. The model has broken to k copies of the A_k balcony model, the chiral theory studied in [8]. The dual gauge group breaks from $SU(3k \frac{(m_f+\tilde{m}_f)}{2} - N_c)$ down to $SU(3 \frac{(m_f+\tilde{m}_f)}{2} - n)^k$, which is consistent with the duality transformation of [8].

It is also straightforward to deform the k -odd theory and flow to the k -even theory. Consider the deformed superpotential

$$W = \sum_{r=2}^{k+1} \frac{s_r}{r} \text{Tr} X^r + \text{Tr} X Y \tilde{Y} - \lambda \text{Tr} X. \quad (7.8)$$

We can consider tuning the couplings s_r such that the conditions for a supersymmetric vacuum become

$$\begin{aligned} (X - a)^{k-1}(X - b) + Y\tilde{Y} &= 0 \\ X\tilde{Y} - \tilde{Y}X^T &= 0 \\ XY - YX^T &= 0. \end{aligned} \tag{7.9}$$

The tracelessness condition permits us to choose as eigenvalues of the $N_c \times N_c$ tensor, X , to be

$$\langle X \rangle = \begin{pmatrix} a_n & 0 \\ 0 & b_m \end{pmatrix} \tag{7.10}$$

where $na + mb = 0$ and where a subscript n indicate that the element is proportional to the $n \times n$ -dimensional unit matrix. We also take $\langle Y \rangle = \langle \tilde{Y} \rangle = 0$. The gauge group breaks from $SU(N_c)$ to $SU(n) \times SU(m) \times U(1)$. The off-diagonal components of X are eaten in the Higgs mechanism leaving adjoint fields X_1 and X_2 charged under their respective gauge groups. The fields Y and \tilde{Y} have massive fields F and \tilde{F} coming from their off-diagonal blocks but massless anti-symmetric fields Y_1 and Y_2 and massless symmetric fields \tilde{Y}_1 and \tilde{Y}_2 coming from their diagonal blocks. We have

$$W = \text{Tr}X_1^k + \text{Tr}X_1Y_1\tilde{Y}_1 + \text{Tr}X_2^2 + \text{Tr}X_2Y_2\tilde{Y}_2. \tag{7.11}$$

Because of the mapping of operators, the dual gauge group breaks similarly from $SU(3k\frac{(m_f+\tilde{m}_f)}{2} - N_c)$ to $SU(3(k-1)\frac{(m_f+\tilde{m}_f)}{2} - n) \times SU(3\frac{(m_f+\tilde{m}_f)}{2} - m)$. We have now two decoupled theories of the same type that we started with. However, X is raised to an odd power in the first vacuum whereas when we started it was raised to an even power.

8. Baryons and Additional Checks on Mezzanine and Balcony Models

An additional interesting check involves certain baryonic flat directions which are present in the D_{k+2} $SU(N_c)$ mezzanine and balcony models, each of which has fields X , Y and \tilde{Y} . As shown in fig. 1, the A_k mezzanine models have baryonic flat directions along which they flow to SO and Sp orchestra models, while the A_k balcony model has baryonic flat directions along which it can flow to the SO and Sp mezzanine models. In the D_k models the situation is different. The D_{k+2} $SU(N_c)$ mezzanine models flow to the A_k mezzanine models with $SO(N_c)$ and $Sp(N_c)$ gauge group, while the D_{k+2} balcony model flows to the A_k orchestra models with $SO(N_c)$ and $Sp(N_c)$ gauge group. The consistency of

the mapping of baryon operators in the D_{k+2} models and their flow to known A_k dualities is a significant non-trivial check on these models. We illustrate this below.

We begin by studying certain baryons in the model of Sec. 6.3. We can form baryons by introducing dressed quarks [4]

$$Q_j = X^j Q; \quad j = 0, \dots, k-1. \quad (8.1)$$

and then contracting the gauge indices on two $SU(N_c)$ epsilon tensors.

$$B_n^{(n_0, n_1, \dots, n_{k-1})} = Y^n Q_0^{n_0} Q_0^{n_0} \dots Q_{k-1}^{n_{k-1}} Q_{k-1}^{n_{k-1}}; \quad n + \sum_{j=0}^{k-1} n_j = N_c \quad (8.2)$$

The total number of baryons is

$$2 \sum_{\{n_j\}} \binom{N_f}{n_0} \dots \binom{N_f}{n_{k-1}} = 2 \binom{kN_f}{N_c - n}. \quad (8.3)$$

We can similarly form anti-baryons.

Under the duality transformation, baryon operators are mapped to other baryon operators in the dual theory. The mapping is

$$B_n^{(n_0, n_1, \dots, n_{k-1})} \leftrightarrow B_{2kN_f+4-n}^{(\bar{n}_0, \bar{n}_1, \dots, \bar{n}_{k-1})}; \quad \bar{n}_j = N_f - n_{k-j-1} \quad (8.4)$$

where in the dual theory the baryons look like

$$B_{2kN_f+4-n}^{(\bar{n}_0, \bar{n}_1, \dots, \bar{n}_{k-1})} = \bar{Y}^{2kN_f+4-n} q_0^{\bar{n}_0} q_0^{\bar{n}_0} \dots q_{k-1}^{\bar{n}_{k-1}} q_{k-1}^{\bar{n}_{k-1}} \quad (8.5)$$

The total number of dual baryons is

$$2 \sum_{\{\bar{n}_j\}} \binom{N_f}{\bar{n}_0} \dots \binom{N_f}{\bar{n}_{k-1}} = 2 \binom{kN_f}{\tilde{N}_c - (2kN_f + 4 - n)} = 2 \binom{kN_f}{kN_f - N_c + n}. \quad (8.6)$$

which is the same as in the electric theory. The fact that this mapping is also consistent with all global symmetries is another non-trivial test of the proposed duality.

We can consider the flat direction along which a particular baryon, $\det Y$, gets a vacuum expectation value. In this direction, $\langle Y_{\alpha\beta} \rangle \propto \delta_{\alpha,\beta}$, where α and β are gauge indices, breaking the group from $SU(N_c)$ to $SO(N_c)$. The adjoint field X splits into fields in the symmetric and the anti-symmetric representations, $X = A + S$. The field \tilde{Y} becomes a field in the symmetric representation of $SO(N_c)$, \hat{Y} . The superpotential gives a mass

to the symmetric fields \hat{Y} and S , while the field A remains massless, with superpotential $W = \text{Tr}A^{k+1}$. The low-energy theory is thus an A_k mezzanine model.

Although we have not worked out the details of the flat direction corresponding to the dual operator, the baryon mapping (8.4) requires that a similar symmetry breaking happen in the dual of the high energy theory, through an expectation value for the operator $\bar{Y}^{2kN_f+4-N_c}q_0^{N_f}q_0^{N_f}\cdots q_{k-1}^{N_f}q_{k-1}^{N_f}$. This suggests that the dual gauge group breaks from $SU(3kN_f+4-N_c)$ to $SO(2kN_f+4-N_c)$ (possibly with other unbroken confining factors) which would correspond to the dual expected for an A_k mezzanine model [7].

We turn next to baryons in the model of Sec. 6.4. As in the case discussed above, we can introduce dressed quarks and form baryons, which in this case require only one epsilon tensor since Y is in the anti-symmetric representation.

$$B_n^{(n_0, n_1, \dots, n_{k-1})} = Y^n Q_0^{n_0} \cdots Q_{k-1}^{n_{k-1}}; \quad 2n + \sum_{j=0}^{k-1} n_j = N_c \quad (8.7)$$

The mapping of baryon operators is

$$B_n^{(n_0, n_1, \dots, n_{k-1})} \leftrightarrow B_{kN_f-2-n}^{(\bar{n}_0, \bar{n}_1, \dots, \bar{n}_{k-1})}; \quad \bar{n}_j = N_f - n_{k-j-1} \quad (8.8)$$

where in the dual theory the baryons look like

$$B_{kN_f-2-n}^{(\bar{n}_0, \bar{n}_1, \dots, \bar{n}_{k-1})} = \bar{Y}^{kN_f-2-n} q_0^{\bar{n}_0} \cdots q_{k-1}^{\bar{n}_{k-1}} \quad (8.9)$$

We can consider the flat direction along which a particular baryon, $\text{Pf } Y$, gets a vacuum expectation value. In this direction, the group is broken from $SU(N_c)$ to $Sp(N_c)$. This time it is the symmetric part of X which is massless, with superpotential $W = \text{Tr}S^{k+1}$. The low-energy theory is thus an A_k mezzanine model. As before, although we have not worked out the details of the flat direction, the baryon mapping (8.8) requires an expectation value for the operator $\bar{Y}^{kN_f-2-N_c}q_0^{N_f}\cdots q_{k-1}^{N_f}$. This suggests that the dual gauge group breaks to $Sp(kN_f-2-N_c)$ as expected for an A_k mezzanine model [7].

We turn next to baryons in the model of Sec. 7. We can form baryons around Y as in Eq. (8.7). The mapping of baryon operators is

$$B_n^{(n_0, n_1, \dots, n_{k-1})} \leftrightarrow B_{\frac{k}{2}(m_f+\tilde{m}_f)-2k-n}^{(\bar{n}_0, \bar{n}_1, \dots, \bar{n}_{k-1})}; \quad \bar{n}_j = m_f - n_{k-j-1} \quad (8.10)$$

We can also form anti-baryons around \tilde{Y} as in Eq. (8.2). The mapping of anti-baryon operators is

$$\tilde{B}_n^{(n_0, n_1, \dots, n_{k-1})} \leftrightarrow \tilde{B}_{k(m_f+\tilde{m}_f)+4k-n}^{(\bar{n}_0, \bar{n}_1, \dots, \bar{n}_{k-1})}; \quad \bar{n}_j = \tilde{m}_f - n_{k-j-1} \quad (8.11)$$

We can consider the flat direction $\langle \text{Pf } Y \rangle$ which breaks the gauge group to $Sp(N_c)$. As before X splits into symmetric and anti-symmetric parts $X = S + A$, with S becoming massive along with \tilde{Y} , leaving the field A with superpotential A^{k+1} . This is an A_k orchestra model. In the dual, the baryon $\bar{Y}^{\frac{k}{2}(m_f + \tilde{m}_f) - 2k - N_c} q_0^{N_f} \cdots q_{k-1}^{N_f}$ gets an expectation value, suggesting that the low-energy gauge group is $Sp(\frac{k}{2}(m_f + \tilde{m}_f) - 2k - N_c)$, which would agree with the A_k orchestra duality transformation.

Similarly, the flat direction $\langle \det \tilde{Y} \rangle$, which breaks the gauge group to $SO(N_c)$, leaves the field S with superpotential S^{k+1} . In the dual, the expectation value for the anti-baryon $\tilde{Y}^{k(m_f + \tilde{m}_f) + 4k - N_c} q_0^{N_f} q_0^{N_f} \cdots q_{k-1}^{N_f} q_{k-1}^{N_f}$ suggests the low-energy gauge group is $SO(k(m_f + \tilde{m}_f) + 4k - N_c)$, which would again agree with the A_k orchestra duality transformation.

9. Summary and Conclusions.

We have presented a number of new examples of duality, which are generalizations of the D_{k+2} models of Ref. [14]. The pattern of new examples resembles the pattern of the A_k models [4,6,7,8] as is clear from comparing fig. 1 and fig. 4. The dualities of the D_{k+2} and A_k models have much in common, such as the fact that they relate pairs of theories of similar type, the presence of gauge singlet mesons in the magnetic superpotentials, and the existence at all self-dual points of marginal operators which take the form of meson mass terms.

However, despite the similarities of these patterns, there are many differences as well, and a number of puzzles remain.

First, although the traditional ADE classification of groups and singularities contains D_{k+2} for all positive k , the present examples of duality seem only to allow k odd, except for the chiral balcony theory where the algebra is different. This is more than just a mathematical oddity; it is a real problem for the duality. Were our understanding of these dualities complete, we would know the mapping of the operator X^{2p+1} to the dual theory. We could then perturb the electric superpotential by X^{2p+1} and the magnetic superpotential by the image of this operator, and the electric theory would then flow to a D_{2p+2} model with the magnetic theory flowing to its dual. We would thereby derive the duality for k even. But this approach fails, showing that we do not understand the mapping for the operators X^{2p+1} . It is especially confusing that this issue does not arise for the balcony D_{k+2} model. Why this theory is fundamentally different is not yet clear to us.

Second, as we have seen, the field Y may be chosen to be any one of the standard two-index tensors of SU , SO , or Sp , but the field X must be in the adjoint of SU , the symmetric tensor of SO , or the anti-symmetric tensor of Sp . These three fields appear in the orchestra models of the A_k series [4,6,8]. For other choices of X the superpotential term XY^2 cannot be written, and no obvious generalization seems to work. Note that in the A_k series many generalizations of the original X^{k+1} superpotential appear in the list of dual models. Why X is so restricted in the D_{k+2} models is unknown.

Third, unlike the A_k series, there do not seem to be any logic models of the form $G \times G'$ which confine to form the models of the D_{k+2} series. We do on the other hand find models of the form of a product of SO or Sp groups which generalize the stage model with $SU \times SU$ gauge group found in Ref. [14]. All the D_{k+2} models with gauge group G flow under certain superpotential perturbations to the stage models with gauge group $G \times G$, except for the balcony model, which flows to copies of itself (with smaller k) under such perturbations.

Fourth, we have not yet found any sign of models corresponding to the singularities for E_6 , E_7 or E_8 . Notice however that $E_4 = A_4$ and $E_5 = D_5$ do appear in our list. The significance of this is again unclear.

Fifth, the patterns of the dual groups as listed on the right-hand side of fig. 1 do not closely resemble those of fig. 4, despite the fact that the A_3 and D_3 cases agree. There are clear patterns in both cases; but why should they be so different from one another? Are there other classes of models which, once discovered, will show that the organizing plan that we have used is misleading?

The more basic issues raised by this work point toward a number of areas for additional research. What role is this matrix generalization of singularity theory playing in duality? How is it connected with the new concepts of matrix geometry which arise in the context of D-branes [17] ? Are there D-brane constructions of these theories in which the duality transformations might be manifest, or if not manifest at least related to known string-associated dualities? Are marginal operators important in understanding the nature of or source of duality, or are they just an interesting sidelight? We expect there will be many interesting discoveries to come.

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